Some Considerations About the Finite Difference Time Domain Method in General Curvilinear Coordinates

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***Abstract*—Some considerations about the stability criterion and the dispersion of the Finite Difference Time Domain Method in General Curvilinear Coordinates are presented. An expression for the stability criterion and for the dispersion relation is found to be possible only for a mesh without curvature, and for this particular case, a dispersion relation is obtained in three dimensions. The analysis of this relation shows that a resolution of 10 cells per wave-length is enough in order to keep the dispersion error within a reasonable limit. A study of the reflection in the interface between two regions with different metrics is presented as well.**

***Index Terms*—General curvilinear coordinates, FDTD.**

I. Introduction

The FDTD method has been successfully applied to the study of electromagnetic wave propagation, and a powerful generalization of this algorithm is the FDTD method in General Curvilinear Coordinates (FDTD-GCC) [1]–[3]. The FDTD-GCC mesh is able to conform to the shape of the structure. The boundary conditions can be applied in a natural, precise, and easy way. Further, we have the freedom to develop a coarse mesh in regions where a soft behavior of the field is expected and a finer mesh in corners and gaps, where high field variations are suspected. Although the stability criterion has been outlined by several authors, so far little has been said about the dispersive properties of this technique and its relative accuracy [4].

In this letter, it will be shown that the stability criterion presented by the bibliography [2], [3], is only valid for a homogeneous region in which the mesh has no curvature, or in other words, for the mesh that keeps the direction of the curvilinear bases constant, that is, with constant skewing. It is a rough approach for the general case in which curvature in the mesh is involved. For the particular case with no curvature, the dispersive relationship for a three-dimensional mesh is derived. The dispersive effects are studied for different propagation directions with respect to the base vectors and for different skewing angles between these base vectors (Fig. 1(a)). As a general FDTD-GCC mesh is a non-uniform mesh with different metric tensors in different regions of the modeling, a brief analysis of the reflection in the transition between two regions with different metrics is also presented.

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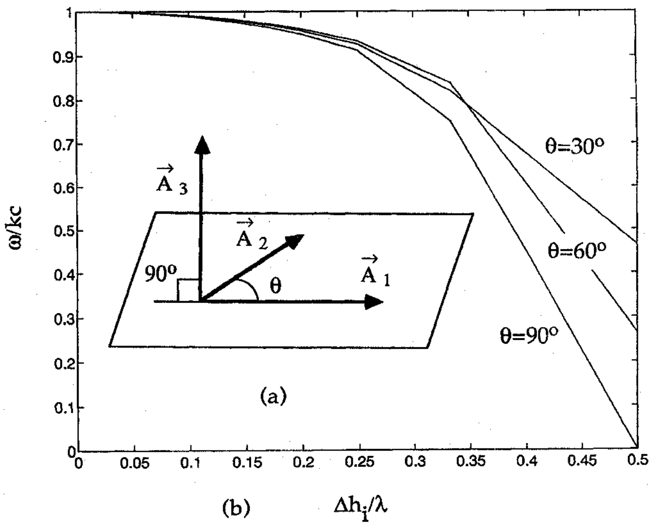

Fig. 1. (a) Curvilinear basis vectors. (b) Normalized phase velocity $\omega/(kc)$ versus normalized cell size $\Delta h_1/\lambda$ for meshes with various skewing angles θ with $\Delta h_1 = \Delta h_2 = \Delta h_3$. Propagation of the wave is along the $\vec{A}_1$ direction with $\alpha = 0$.

II. The Dispersion Characteristics of FDTD-GCC

It is assumed that any electromagnetic field can be decomposed as a linear combination of plane monochromatic waves. Therefore, the dispersion of the FDTD-GCC method is analyzed by assuming a plane, monochromatic traveling wave as a solution of the discretized vector wave equation. We assume the following solution: $\vec{E}(\vec{u}) = \exp(\omega t - j\vec{k}\cdot\vec{u})\vec{e}$. That is, a plane, monochromatic traveling wave in which $\vec{u} = (u^1, u^2, u^3)$ are the generalized curvilinear coordinates, $\vec{e} = \sum_{l=1}^{3} \alpha^l \vec{A}_l$ is the polarization vector, and $\{\vec{A}_l\}_{l=1}^{3}$ are the curvilinear bases.

In a general curvilinear coordinate system, the dual bases $\{\vec{A}^l\}_{l=1}^{3}$ are defined as well so that $\vec{A}^i \cdot \vec{A}_l = \delta_l^i$. The gradient operator can be written as,

$$\nabla = \vec{A}^1 \frac{\partial}{\partial u^1} + \vec{A}^2 \frac{\partial}{\partial u^2} + \vec{A}^3 \frac{\partial}{\partial u^3}, \tag{1}$$

and the application of ∇ over the plane wave using central differencing and assuming that the increment in the general

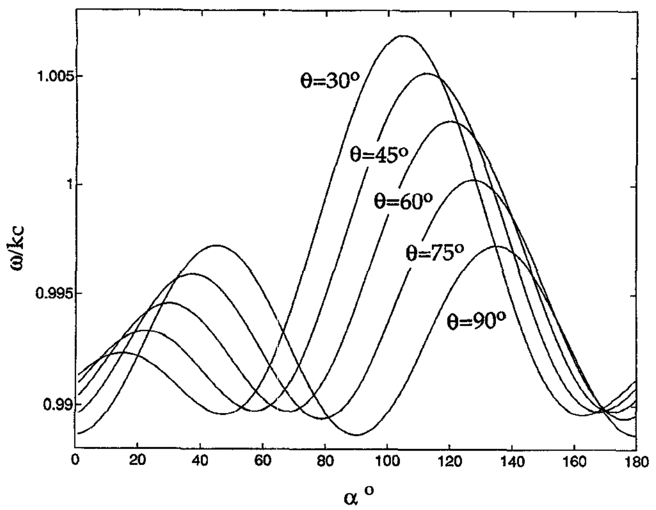

Fig. 2. Normalized phase velocity $\omega/(kc)$ versus propagation angle α for various skewing angles θ of the grid with $\Delta h_1/\lambda = 0.1$ and with $\Delta h_1 = \Delta h_2 = \Delta h_3$.

curvilinear coordinates is $\Delta u^i = 1$, can be expressed by,

$$\nabla = -2j \sum_{i=1}^{3} \vec{A}^i \sin\left(\frac{\Delta(k_i u^i)}{2}\right), \tag{2}$$

and the divergence of the electric field becomes,

$$\nabla \cdot \vec{E} = -2j \sum_{i=1}^{3} \alpha^i \sin\left(\frac{\Delta(k_i u^i)}{2}\right) \cdot \exp(\omega t - j\vec{k} \cdot \vec{u}). \tag{3}$$

For a region in which the direction of the bases vectors is constant, setting $\Delta u^i = 1$, we have $\Delta(k_i u^i) = k_i$. As long as we have a plane wave, $\vec{e} \cdot \vec{k} = 0$, and we obtain the condition $\nabla \cdot \vec{E} = 0$. From this fact, it is possible to derive the stability criterion given by previous authors [2], [3].

If we have a general curvilinear mesh in which the orientation of the bases change from point to point, which is the general case, then $\Delta(k_i u^i) = k_i + u^i \Delta k_i$. The second term on the right of the previous equation is a nonphysical term due to the curvature of the mesh. Thus, $\nabla \cdot \vec{E} = 0$ is not guaranteed to be true. This is an important fact that must be kept in mind when using the FDTD method in General Curvilinear Coordinates. The divergence free conditions for the fields can not be numerically achieved. However, the stability criterion given in, [2], [3], although not exact, can be useful as a reference.

For the particular case in which the mesh is without curvature, the stability criterion is exact and for a region in which the metric tensor is constant, $g^{ij} = \vec{A}^i \cdot \vec{A}^j = \text{constant}^{ij}$, it is possible to derive a numerical dispersive relation.

Using central differencing for the time derivative as well, and substituting (2) into the vector wave equation, we obtain the numerical three-dimensional dispersive relation for a mesh having constant g^{ij},

$$\frac{\sin^2(\frac{1}{2}\omega\Delta t)}{c^2\Delta t^2} = \sum_{i=1,j=1}^{3,3} g^{ij} \sin\left(\frac{k_i}{2}\right) \sin\left(\frac{k_j}{2}\right). \tag{4}$$

This is an implicit function of $\vec{k}$, where $\vec{k}$ can be solved numerically for each ω.

III. Numerical Analysis

In order to simplify the analysis, we will assume a region of the mesh with constant g^{ij}. The cell dimensions are $\Delta h_1, \Delta h_2, \Delta h_3$, and the vectors are shown in Fig. 1(a). $\vec{A}_3$ is orthogonal to the plane defined by $\vec{A}_1$ and $\vec{A}_2$ and θ is the angle between these two vectors. In that region we have, $g_{11} = \Delta h_1^2$, $g_{12} = \Delta h_1 \Delta h_2 \cos(\theta)$, $g_{13} = 0$, $g_{22} = \Delta h_2^2$, $g_{23} = 0$, $g_{33} = \Delta h_3^2$, where $g_{ij} = \vec{A}_i \cdot \vec{A}_j$. From these, and assuming a plane, monochromatic traveling wave that propagates with angle α with respect to $\vec{A}_1$ in the plane defined by $\vec{A}_1$ and $\vec{A}_2$ and polarized in the direction of $\vec{A}_3$, (4) becomes,

$$\begin{aligned}\frac{\sin^2(\frac{1}{2}\omega\Delta t)}{c^2\Delta t^2} &= \frac{1}{\Delta h_1^2 \sin^2\theta} \sin^2\left(\frac{k\Delta h_1}{2}\cos\alpha\right) \\ &\quad - 2\frac{\cos\theta}{\Delta h_1 \Delta h_2 \sin^2\theta} \sin\left(\frac{k\Delta h_2}{2}\cos(\alpha-\theta)\right) \\ &\quad \times \sin\left(\frac{k\Delta h_1}{2}\cos\alpha\right) \\ &\quad + \frac{1}{\Delta h_2^2 \sin^2\theta} \sin^2\left(\frac{k\Delta h_2}{2}\cos(\alpha-\theta)\right).\end{aligned} \tag{5}$$

This equation reduces to the standard dispersion relation for a conventional orthogonal Cartesian mesh, [5], with $\theta = 90°$. The above expression is also very similar to the one presented by Ray [4] for a particular two-dimensional case, but there is a slight difference of a factor of two ([4], (3)) that may be due to some mistake in the derivation.

Equation (5) is first solved in order to obtain k for the particular case of Fig. 1(a) as a function of the mesh resolution in terms of the wavelength. This is made for several angles of skewing θ between $\vec{A}_1$ and $\vec{A}_2$ and with a constant angle of $\alpha = 0$ of the incident wave. The cell dimensions are set to $\Delta h_1 = \Delta h_2 = \Delta h_3$ and the time step is chosen to be the maximum allowed by the stability criterion, [3], $\Delta t = 1/(c\sqrt{g^{11} + 2g^{12} + g^{22} + g^{33}})$.

The normalized phase velocity $\omega/(kc)$ is presented as a function of the wavelength resolution in Fig. 1(b). An ideal algorithm would have $\omega/(kc) = 1$, however the presented results show that the dispersion error can be kept under a reasonable limit for a resolution of the mesh in terms of the wavelength of $\Delta h_1 = \Delta h_2 \leq \lambda/10$.

Equation (5) is solved again in order to obtain k for several angles θ between $\vec{A}_1$ and $\vec{A}_2$ as a function of the angle α of the incident wave. The cell size is set to $\Delta h_1 = \Delta h_2 = \Delta h_3 = \lambda/10$, and the time step is chosen in the same way as the previous case. The normalized numerical phase velocity is presented in Fig. 2. The presented results show that the dispersion error is small enough for the resolution of the mesh, and it is observed that the numerical phase velocity is always maximum in the oblique direction to the cell.

Comparisons made by authors having solved the dispersive relation for smaller Δt than the maximum allowed by the

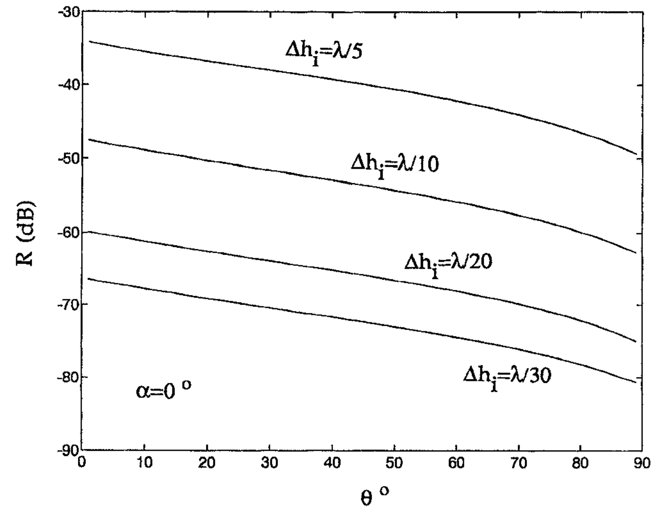

Fig. 3. Reflection in the transition from a mesh without skewing to a skewed mesh, both with $\Delta h_1 = \Delta h_2 = \Delta h_3$ for different angles in the skewing of the second mesh. $\vec{A}_1$ and $\vec{A}_3$ are the same in both meshes, and $\vec{A}_2$ is rotated by θ degrees in the second mesh with respect to the first one.

stability criterion, shows that the dispersion error increases with decreasing Δt. These results are not shown in this letter.

Once the phase velocity is known for each angle of incidence and skewing of the mesh, it is possible to calculate the reflection in the interface between two regions that use a different mesh. The reflection coefficient is derived from the Fresnel equations [6]. The reflection in the transition from a mesh without skewing to a skewed mesh with both having the same space increments of $\Delta h_1 = \Delta h_2 = \Delta h_3$ is presented in Fig. 3 for different angles θ in the skewing of the second mesh. The vectors $\vec{A}_1$ and $\vec{A}_3$ have the same direction in both meshes while the orientation of $\vec{A}_2$ is changed in the second mesh with respect to the first one. The reflection is found to be negligible even for a very rough mesh with $\Delta h_1 = \Delta h_2 = \Delta h_3 = \lambda/5$.

IV. Conclusion

Some considerations were made about the stability condition, and the dispersion relation in the Finite Difference Time Domain algorithm in General Curvilinear Coordinates. It was shown that a general analytical expression for both is possible only for a mesh without curvature. For this specific case, and for a constant metric tensor, a three-dimensional dispersive relation was obtained. Moreover, this particular case was analyzed, and it was found that the error in the phase velocity can be kept under a reasonable limit with a resolution above 10 cells per wavelength. The reflection in the interface between two regions with a different metric was studied as well and the value of the reflection coefficient was found to be negligible.

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